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LETTER TO THE EDITOR

The NLS$^-$ equation and its $SL(2, R)$ structure

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Abstract. The relationship of the group $SL(2, R)$ and the NLS$^-$ equation is presented. As a consequence, the $SL(2, R)$ gauge equivalence between the NLS$^-$ and the M-HF model is proved, which provides a new example in geometrically explaining dynamical properties of soliton equations by the $SL(2, R)$ structure.

1. Introduction

Much evidence has reflected the importance of the relationship between the nonlinear partial differential equations which can be solved by the inverse scattering method (such as SG, KdV and MKdV) and the group $SL(2, R)$ of 2×2 real matrices with determinant 1 [1–4]. The indication that there is such a relationship is the existence of three 1-forms $\{\omega_1, \omega_2, \omega_3\}$ on R^2 (the space of the independent variables x and t of the differential equation), whose coefficients depend only on u (the dependent variable) and its derivatives, such that they satisfy the structure equations of a pseudospherical surface (see [3,4]):

$$\mathrm{d}\omega_1 = \omega_3 \wedge \omega_2 \qquad \mathrm{d}\omega_2 = \omega_1 \wedge \omega_3 \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \omega_2. \tag{1}$$

Such equations are also known as differential equations describing pseudospherical surfaces (PSS) or, in other words, PSS equations (see [4–6]). It is well known that the SG, KdV, MKdV, Burgers' equation, etc. describe PSS (see [3–6]).

One can verify straightforwardly that (1) is equivalent to saying that

$$\mathrm{d}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \Omega \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix} \in sl(2, R) \tag{2}$$

is a completely integrable system, i.e. $\mathrm{d}\Omega - \Omega \wedge \Omega = 0$. The $sl(2, R)$-valued 1-form Ω may be regarded as defining a connection on a principal $SL(2, R)$ bundle over R^2 and the soliton equation expresses the fact that the curvature $F = \mathrm{d}\Omega - \Omega \wedge \Omega$ of this connection vanishes. Such dynamical properties as the existence of an infinite number of conservation laws and symmetries, and the Bäcklund transformations for PSS equations can be geometrically explained by use of relations (1), i.e. their $SL(2, R)$ structure (see [2–6]). Indeed, it seems likely that the group $SL(2, R)$ is the key to understanding the integrability of these soliton equations.

On the other hand, the nonlinear Schrödinger equation (NLS): $\mathrm{i}q_t + q_{xx} + 2\kappa|q|^2 q = 0$, where κ is a real constant distinguishing the equation between attractive ($\kappa > 0$) and repulsive

($\kappa < 0$) type, is a representative example in the theory of integrable systems [7]. The NLS models a wide range of physical phenomena (one may refer to [8] for a catalogue of its physical applications). Therefore, the NLS has received a systematical study in recent decades (see, for examples, [7,9]). As usual, we denote by NLS^+ and NLS^- the NLS with $\kappa = 1$ and -1, respectively. To the author's best knowledge, the NLS^+ is realized to be related to the group $SU(2)$ and while the NLS^- to the group $SU(1, 1)$ in the existing literature. To our surprise, the NLS^- fits into the $SL(2, R)$ picture, which is what we shall display in this Letter.

2. $SL(2, R)$ structure

Since the NLS^- has no light-soliton but dark-soliton solutions, we put $q = r\mathrm{e}^{-2\mathrm{i}\rho^2 t}$, where ρ is a positive constant, and get an equivalent equation for r which reads

$$\mathrm{i}r_t + r_{xx} - 2(|r|^2 - \rho^2)r = 0. \tag{3}$$

As pointed out in [7], we need to add the finite density boundary condition, i.e. $r \to \rho$ as $x \to +\infty$ and $r \to \rho\mathrm{e}^{\mathrm{i}2\beta}$ as $x \to -\infty$ (where β is a real constant), in solving (3). Now we convert (3) into the real form ($r(x, t) = u(x, t) + \mathrm{i}v(x, t)$):

$$u_t + v_{xx} - 2(u^2 + v^2 - \rho^2)v = 0 \qquad - v_t + u_{xx} - 2(u^2 + v^2 - \rho^2)u = 0. \tag{4}$$

One may verify directly that system (4) describes PSS with

$$\begin{aligned}
\omega_1 &= 2u dx + (4\lambda u - 2v_x)\mathrm{d}t \\
\omega_2 &= -2v dx - (4\lambda v + 2u_x)\mathrm{d}t \\
\omega_3 &= -2\lambda dx - [4\lambda^2 + 2(u^2 + v^2 - \rho^2)]\mathrm{d}t
\end{aligned}$$

(λ is a spectral parameter) and hence admits the $SL(2, R)$ structure as illustrated in section 1. The corresponding connection 1-form in (2) is as follows:

$$\begin{aligned}
\widetilde{\Omega} =& \ \{-\lambda\sigma_2 + U\}\,\mathrm{d}x + \left\{-2\lambda^2\sigma_2 + 2\lambda U - (U^2 - \rho^2 + U_x)\sigma_2\right\}\mathrm{d}t \\
:=& \ \widetilde{L}(x, t, \lambda)\mathrm{d}x + \widetilde{M}(x, t, \lambda)\mathrm{d}t
\end{aligned} \tag{5}$$

in which

$$\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} -v & u \\ u & v \end{pmatrix} \in sl(2, R).$$

Although the NLS^- equation and other soliton equations such as the SG, KdV, etc. have some different physical characteristics (for example, the SG and KdV have light-soliton solutions, but the NLS^- has dark-soliton solutions and no light-soliton solutions), the above result shows that they have the same $SL(2, R)$ structure. Thus the dynamical properties of the NLS^-, like the existence of infinite number of conservation laws and symmetries, can also be geometrically interpreted in the same way as those of the SG, KdV, MKdV and Burgers' equation from the viewpoint of the $SL(2, R)$ structure.

3. $SL(2, R)$ gauge equivalence

The $SU(1, 1)$ gauge equivalence between the NLS^- and the M-HF model was displayed in [10], which gives a dual geometric interpretation of the well known fact that there is an $SU(2)$ gauge equivalence between the NLS^+ and the HF model [11]. Now, based on the fact displayed in section 2, we are specially interested in whether there exists an $SL(2, R)$ gauge transformation between the NLS^- and the M-HF model. In the present section we shall give an affirmative answer to this problem.

Before proving the conclusion, let us give a brief description of the M-HF model—the Schrödinger flow of maps into $H^2 \hookrightarrow R^{2+1}$ (see [10] for details),

$$\boldsymbol{S}_t = \boldsymbol{S}\dot{\times}\boldsymbol{S}_{xx} \tag{6}$$

where $\boldsymbol{S} = (s_1(x,t), s_2(x,t), s_3(x,t)) \in R^{2+1}$ satisfies $s_1^2 + s_2^2 - s_3^2 = -1$ with $s_3 > 0$, and $\dot{\times}$ denotes the pseudo cross product, i.e. for arbitrary two vectors $\boldsymbol{a}, \boldsymbol{b} \in R^{2+1}$, $\boldsymbol{a}\dot{\times}\boldsymbol{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, -(a_1b_2 - a_2b_1))$. System (6) also describes PSS with

$$\begin{aligned}
\omega_1 &= 2\lambda s_2 \mathrm{d}x + (4\lambda^2 s_2 - 2\lambda s_1 s_{3x} + 2\lambda s_3 s_{1x})\mathrm{d}t \\
\omega_2 &= 2\lambda s_1 \mathrm{d}x + (4\lambda^2 s_1 + 2\lambda s_2 s_{3x} - 2\lambda s_3 s_{2x})\mathrm{d}t \\
\omega_3 &= -2\lambda s_3 \mathrm{d}x + (-4\lambda^2 s_3 - 2\lambda s_2 s_{1x} + 2\lambda s_1 s_{2x})\mathrm{d}t
\end{aligned}$$

where λ is a spectral parameter. The corresponding connection 1-form Ω in (2) is

$$\Omega = \lambda Q \mathrm{d}x + \left\{2\lambda^2 Q - \lambda Q Q_x\right\}\mathrm{d}t := L(x,t,\lambda)\mathrm{d}x + M(x,t,\lambda)\mathrm{d}t \tag{7}$$

in which

$$Q = \begin{pmatrix} s_1 & s_2 + s_3 \\ s_2 - s_3 & -s_1 \end{pmatrix} \in sl(2, R)$$

with $Q^2 = -I$. It is obivous that $\boldsymbol{S} = (s_1, s_2, s_3)$ and Q are determined by each other.

Now we are in a position to prove that there is an $SL(2, R)$ gauge transformation between the NLS$^-$ (4) and the M-HF model (6).

First, suppose that (u, v) is a solution to the NLS$^-$ (4) satisfying the finite density boundary condition. The corresponding connection 1-form $\widetilde{\Omega}$ is denoted as in (5). We consider the following $SL(2, R)$ gauge transformation:

$$\Omega \to \widetilde{\Omega} = \mathrm{d}AA^{-1} + A\Omega A^{-1} \tag{8}$$

or equivalently

$$\widetilde{L}(x,t,\lambda) = A(x,t)L(x,t,\lambda)A(x,t)^{-1} + A_x(x,t)A(x,t)^{-1} \tag{9}$$

$$\widetilde{M}(x,t,\lambda) = A(x,t)M(x,t,\lambda)A(x,t)^{-1} + A_t(x,t)A(x,t)^{-1} \tag{10}$$

where $A(x,t)$ is an $SL(2, R)$ matrix which will be determined later. We hope to prove that Ω is exactly a connection 1-form (7) after A and Q have been suitably chosen. In fact, substituting $\widetilde{L}(x,t,\lambda) = -\lambda\sigma_2 + U$ and $L(x,t,\lambda) = \lambda Q(x,t)$ into (9) and comparing the coefficients of λ in the equation, we obtain

$$\sigma_2 = -A(x,t)QA(x,t)^{-1} \qquad \text{i.e.} \quad Q = -A(x,t)^{-1}\sigma_2 A(x,t) \tag{11}$$

$$U(x,t) = A_x(x,t)A(x,t)^{-1} \qquad \text{i.e.} \quad \partial_x A(x,t) = U(x,t)A(x,t). \tag{12}$$

Furthermore, substituting $\widetilde{M} = -2\lambda^2\sigma_2 + 2\lambda U - (U^2 - \rho^2 + U_x)\sigma_2$ and $M(x,t,\lambda) = 2\lambda^2 Q(x,t) - \lambda Q(x,t)Q_x(x,t)$ into (10), the constant term leads to

$$\partial_t A(x,t) = -\left(U^2 - \rho^2 + U_x\right)\sigma_2 A(x,t). \tag{13}$$

Combining (13) with (12), we see that we should make a choice of $A(x,t)$ to be a fundamental solution to (2) with the Ω being given by $\widetilde{\Omega}$ at $\lambda = 0$. Then, the other two coefficient terms of λ^2 and λ in (10) are

$$-2\lambda^2\sigma_2 + 2\lambda U(x,t) = A(x,t)\{2\lambda^2 Q - \lambda Q Q_x\}A(x,t)^{-1}. \tag{14}$$

Because Q is given by (11), we see that the coefficients of λ^2 in the right- and left-hand sides of (14) coincide automatically. So what remains to show is that the coefficients of λ in the right- and left-hand sides of (14) are the same, i.e.

$$QQ_x = -A(x,t)^{-1}2U(x,t)A(x,t). \tag{15}$$

Indeed, from (11), we have

$$Q_x = -A^{-1}\sigma_2 A_x + A^{-1}A_x A^{-1}\sigma_2 A = QA^{-1}A_x + A^{-1}A_x A^{-1}\sigma_2 A \tag{16}$$

and, from the skew commutivity of σ_2 and U, we have

$$A^{-1}A_x A^{-1}\sigma_2 A = A^{-1}U\sigma_2 A = -A^{-1}\sigma_2 UA = QA^{-1}A_x. \tag{17}$$

Thus, by substituting (17) into (16), we see $Q_x = 2QA^{-1}A_x = QA^{-1}2UA$. Multiplying both sides of this equation by Q and using $Q^2 = -I$, we arrive at the desired identity (15). This proves that the connection 1-form Ω (resp. Q) transformed from the connection 1-form $\widetilde{\Omega}$ of the NLS$^-$ (4) by the gauge transformation A is a connection 1-form (resp. a solution) of the M-HF model (6).

Next, we shall prove that the above process is in fact reversible. Suppose that $Q(x,t)$ with $Q^2 = -I$ is a solution to (6). We want to choose an $SL(2,R)$ matrix $A(x,t)$ such that $\det A = 1$, $\sigma_2 = -AQA^{-1}$ and

$$A_x A^{-1} = \begin{pmatrix} -v & u \\ u & v \end{pmatrix} := U \tag{18}$$

for some functions u and v. Indeed, the general solutions to $\sigma_2 = -AQA^{-1}$ and $\det A = 1$ are of the form:

$$A = \Phi_{\theta(x,t)}A_0 \qquad \text{in which } \theta = \theta(x,t) \text{ is an arbitrary function}$$

$$A_0 = \frac{1}{\sqrt{2(s_3+1)}}\begin{pmatrix} s_1 & s_2 - s_3 - 1 \\ s_2 + s_3 + 1 & -s_1 \end{pmatrix}$$

and

$$\Phi_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

From this, it is easy to see that we can choose a suitable θ such that (18) is satisfied. Now put $L^A = A_x A^{-1} + ALA^{-1} = -\lambda\sigma_2 + U$ and $M^A = A_t A^{-1} + AMA^{-1} = -2\lambda^2\sigma_2 + 2\lambda U + A_t A^{-1}$ with $L = \lambda Q$ and $M = 2\lambda^2 Q - \lambda QQ_x$. Since Q satisfies the integrability condition, $\mathrm{d}\Omega - \Omega \wedge \Omega = 0$ or equivalently $L_t - M_x + [L, M] = 0$, we have

$$L_t^A - M_x^A + [L^A, M^A] = 0. \tag{19}$$

Assume $A_t A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ for some functions α, β and γ. The vanishing of the coefficient of λ in (19) leads to $\alpha = -u_x$ and $\beta = -\gamma - 2v_x$. Then, substituting these relations into (19), the identification of the two off-diagonal entries of the constant term produces $\gamma = -v_x - 2(u^2 + v^2) + \tau(t)$ for some function $\tau(t)$. Hence, we have

$$A_t A^{-1} = -(U^2 + U_x)\sigma_2 + \tau(t)\sigma_2. \tag{20}$$

Note again that the above restriction on A allows an arbitrariness in A of the form $A \to \Phi_{\sigma(t)}A := \widetilde{A}$ for an arbitrary function $\sigma(t)$. In fact, if we denote

$$\widetilde{U} = \begin{pmatrix} -\widetilde{v} & \widetilde{u} \\ \widetilde{u} & \widetilde{v} \end{pmatrix} := \widetilde{A}_x \widetilde{A}^{-1}$$

under this transformation, then a direct computation shows

$$\widetilde{A}_t \widetilde{A}^{-1} = -\left(\widetilde{U}^2 + \widetilde{U}_x\right)\sigma_2 + \left(\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} + \tau(t)\right)\sigma_2.$$

So if we require $\sigma(t)$ to satisfy $\mathrm{d}\sigma(t)/\mathrm{d}t = -\tau(t) + \rho^2$, then A can be modified so that for the new A the second term on the right-hand side of (20) is $\rho^2\sigma_2$. This implies that M^A is the same as $\widetilde{M}$ in (5) and hence the 1-form $\widetilde{\Omega}$ (resp. (u, v)) constructed from the Ω is a connection 1-form (resp. a solution) of the NLS$^-$ (4). The proof of the existence of an $SL(2, R)$ gauge equivalence between the NLS$^-$ equation and the M-HF model is completed.

Finally, we remark that the above gauge transformation can be geometrically interpreted as a map that maps a family of PSS determining the NLS$^-$ into another one determining the M-HF model. This provides a new example in geometrically explaining dynamical properties of soliton equations by the $SL(2, R)$ structure.

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